

Extended modules and Ore extensions

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Abstract

In this paper we investigate extended modules for a special class of Ore extensions. We will assume that R is a ring and A will denote the Ore extension $A := R[x_1, \dots, x_n; \sigma]$ for which σ is an automorphism of R , $x_i x_j = x_j x_i$ and $x_i r = \sigma(r) x_i$, for every $1 \leq i, j \leq n$. With some extra conditions over the ring R , we will prove Vaserstein's, Quillen's patching, Horrocks' and Quillen-Suslin's theorems for this type of non-commutative rings.

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1 Introduction

The study of finitely generated projective modules over a ring B induces the notions of \mathcal{PSF} , \mathcal{PF} , Hermite (\mathcal{H}), d -Hermite rings, and many other classes of interesting rings. B is \mathcal{PSF} if any finitely generated projective left B -module is stably free; we say that B is \mathcal{PF} if any finitely generated projective left B -module is free; B is Hermite (\mathcal{H}) if any stably free left B -module is free. The ring B is d -Hermite if any stably free left B -module of rank $\geq d$ is free. Note that $\mathcal{H} \cap \mathcal{PSF} = \mathcal{PF}$. In this paper we will study the class of extended modules which is also very useful for the investigation of projective modules. This special class arises when we try to generalize the famous Quillen-Suslin theorem about projective modules over polynomial rings with coefficients in $PIDs$ to a wider classes of coefficients, or yet to the non-commutative rings of polynomial type (see [1], [2], [3] and [7]). We are interested in investigating extended modules and rings for some special classes of Ore extensions. Thus, if nothing contrary is assumed, we will suppose that R is a ring and A will denote the Ore extension $A := R[x_1, \dots, x_n; \sigma]$ for which σ is an automorphism of R , $x_i x_j = x_j x_i$ and $x_i r = \sigma(r) x_i$, for every $1 \leq i, j \leq n$. In some places we will assume some extra conditions on R .

Some notation is needed as well as to recall some definitions and basic facts. $\langle X \rangle$ will denote the two-sided ideal of A generated by x_1, \dots, x_n . Often we will use also the following notation for A , $A = \sigma(R)\langle X \rangle$. An element $p = c_0 + c_1 X_1 + \dots + c_t X_t \in A$, with $c_0, c_i \in R$ and $X_i \in \text{Mon}(A)$, $1 \leq i \leq t$, will be denoted also as $p = p(X)$, where $\text{Mon}(A) := \{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$. The elements of $\text{Mon}(A)$ will be represented by x^α or in capital letters, i.e., $x^\alpha = X$. All modules are left modules

if nothing contrary is assumed. We will use the left notation for homomorphisms and row notation for matrix representation of homomorphisms between free modules (see Remark 1 in [5]).

Definition 1.1. *Let $T \supseteq S$ be rings.*

- (i) *Let M be a T -module, M is extended from S if there exists an S -module M_0 such that $M \cong T \otimes_S M_0$. It also says that M is an extension of M_0 with respect to S .*
- (ii) *$\mathfrak{M}(T)$ denotes the family of finitely generated T -modules, $\mathfrak{P}(T)$ the family of projective T -modules in $\mathfrak{M}(T)$ and $\mathfrak{P}^S(T)$ the family of modules in $\mathfrak{P}(T)$ extended from S .*
- (iii) *The ring T is extended with respect to S , also called S -extended, if every finitely generated projective T -module is extended from S , i.e., $\mathfrak{P}(T) \subseteq \mathfrak{P}^S(T)$. For the Ore extension $A := R[x_1, \dots, x_n; \sigma]$, we will say that A is \mathcal{E} if A is R -extended.*

Proposition 1.2 (Bass's Theorem). *Let I be a two-sided ideal of a ring R such that $I \subseteq \text{Rad}(R)$, and let P, Q be projective R -modules. Then, $P \cong Q$ if and only if $P/IP \cong Q/IQ$ as R/I -modules. In particular, P is R -free if and only if P/IP is R/I -free.*

Proof. See [4], Lemma 2.4. □

Proposition 1.3. *Let $T \supseteq S \supseteq R$ be rings and M a T -module.*

- (i) *If M is an extension of M_0 with respect to S and M_0 is an extension of L_0 with respect to R , then M is an extension of L_0 with respect to R .*
- (ii) *If T is R -extended, then T is S -extended.*
- (iii) *Let I be a proper two-sided ideal of T , with $S \cong T/I$. If T is \mathcal{PF} then S is \mathcal{PF} .*
- (iv) *Let J be a two-sided ideal of R such that $J \subseteq \text{Rad}(R)$. If R/J is \mathcal{PF} , then R is \mathcal{PF} .*

Proof. (i) $M \cong T \otimes_S M_0$, $M_0 \cong S \otimes_R L_0$, then $M \cong T \otimes_S S \otimes_R L_0 \cong T \otimes_R L_0$.

(ii) $M \cong T \otimes_R M_0 \cong (T \otimes_S S) \otimes_R M_0 \cong T \otimes_S (S \otimes_R M_0)$.

(iii) Let $M \in \mathfrak{P}(S)$. Then, $M \oplus M' \cong S^r$ for some S -module M' , therefore $(T \otimes_S M) \oplus M'' \cong T^r$, i.e., $T \otimes_S M \in \mathfrak{P}(T)$, so $T \otimes_S M$ is T -free, and hence $T \otimes_S M \cong T^\ell \cong T \otimes_S S^\ell$. Whence, $M \cong S \otimes_S M \cong (T/I \otimes_T T) \otimes_S M \cong T/I \otimes_T (T \otimes_S M) \cong T/I \otimes_T (T \otimes_S S^\ell) \cong (T/I \otimes_T T) \otimes_S S^\ell \cong S \otimes_S S^\ell \cong S^\ell$. Therefore, S is \mathcal{PF} .

(iv) Let $M \in \mathfrak{P}(R)$, then $R/J \otimes_R M \in \mathfrak{P}(R/J)$ so that

$$M/JM \cong R/J \otimes_R M \cong (R/J)^n \cong R/J \otimes_R R^n \cong R^n/JR^n.$$

From Proposition 1.2, $M \cong R^n$, and hence R is \mathcal{PF} . □

2 Extended rings and Ore extensions

From now on in the present paper (except in the last section) we will assume that R is a commutative ring and A will denote the Ore extension $A := R[x_1, \dots, x_n; \sigma]$ for which σ is an automorphism of R , $x_i x_j = x_j x_i$ and $x_i r = \sigma(r) x_i$, for every $1 \leq i, j \leq n$.

Proposition 2.1. *Let M be an A -module. Then,*

- (i) *If M is free, then M is extended from R .*

(ii) If M is an extension of M_0 with respect to R , then

$$M_0 \cong M/\langle X \rangle M. \quad (2.1)$$

Moreover, if M is finitely generated (projective, stably free) as A -module, then M_0 is finitely generated (projective, stably free) as R -module.

Proof. (i) $M \cong A^{(Y)}$, then $M \cong A \otimes_R R^{(Y)}$ (note that this property is still valid for any couple of rings $A \supseteq R$).

(ii) First note that $A/\langle X \rangle \cong R$; if $M \cong A \otimes_R M_0$ then $A/\langle X \rangle \otimes_A M \cong A/\langle X \rangle \otimes_A A \otimes_R M_0$, i.e., $M/\langle X \rangle M \cong A/\langle X \rangle \otimes_R M_0 \cong R \otimes_R M_0 \cong M_0$.

Let $M = \langle \underline{z_1, \dots, z_t} \rangle$ and $w \in M_0$, then $w = \bar{z}$ with $z \in M$; there exist $p_1(X), \dots, p_t(X) \in A$ such that $w = \bar{z} = p_1(X)z_1 + \dots + p_t(X)z_t = p_{01}\bar{z}_1 + \dots + p_{0t}\bar{z}_t$, where p_{0i} is the independent term of $p_i(X)$, $1 \leq i \leq t$. Hence, $M_0 = \langle \bar{z}_1, \dots, \bar{z}_t \rangle$.

If M is projective, then $M \oplus M' = A^{(Y)}$, and $A/\langle X \rangle \otimes_A (M \oplus M') \cong A/\langle X \rangle \otimes_A A^{(Y)}$, i.e., $M_0 \oplus M'/\langle X \rangle M' \cong R^{(Y)}$.

If M is stably free, then $M \oplus A^r = A^s$, so applying $A/\langle X \rangle \otimes_A$ we get $M_0 \oplus R^r \cong R^s$. \square

We can give a matrix description of extended modules and rings. For this we firstly we recall the definition of square similar matrices.

Definition 2.2. Let S be a ring and $F, G \in M_s(S)$, it says that F and G are similar if there exists $P \in G_s(S)$ such that $F = PGP^{-1}$. In particular, let $F(X)$ be a square matrix over A of size $s \times s$ and $F(0)$ the matrix over R obtained from $F(X)$ replacing all the variables x_1, \dots, x_n by 0,

$$F(X) \approx F(0) \Leftrightarrow F(0) = P(X)F(X)P(X)^{-1}, \text{ for some } P(X) \in GL_s(A). \quad (2.2)$$

Theorem 2.3. Let M be a finitely generated projective A -module and $F(X) \in M_s(A)$ be an idempotent matrix such that $M = \langle F(X) \rangle$, where $\langle F(X) \rangle$ is the A -module generated by the rows of $F(X)$.

- (i) If $F(X) \approx F(0)$, then M is extended from R .
- (ii) If M is extended from R , then there exists a non zero matrix $P(X) \in M_s(A)$ such that $P(X)F(0) = F(X)P(X)$.
- (iii) If A is such that for every $s \geq 1$, given an idempotent matrix $F(X) \in M_s(A)$, $F(X) \approx F(0)$, then A is \mathcal{E} .
- (iv) If A is \mathcal{E} , then for each $s \geq 1$, given an idempotent matrix $F(X) \in M_s(A)$, there exists a non zero matrix $P(X) \in M_s(A)$ such that $P(X)F(X) = F(0)P(X)$.

Proof. (i) There exists $P(X) \in GL_s(A)$ such that $P(X)F(X)P(X)^{-1} = F(0)$. Since A is quasi-commutative, $F(0) \in M_s(R)$ is idempotent. Let $M_0 := \langle F(0) \rangle$ the R -module generated by the rows of $F(0)$, then M_0 is a finitely generated projective R -module. We will prove that $\langle F(X) \rangle \cong A \otimes_R M_0$, i.e., M is extended from R . $F(X), P(X)$ define A -endomorphisms of A^s , with $P(X)$ bijective, and $F(0)$ define a R -endomorphism of R^s . Let $G(X) := i_A \otimes_R F(0)$, then the following diagram

$$\begin{array}{ccc} A^s \cong A \otimes_R R^s & \xrightarrow{G(X)} & A \otimes_R R^s \cong A^s \\ P(X) \downarrow & & \downarrow P(X) \\ A^s & \xrightarrow{F(X)} & A^s. \end{array}$$

is commutative since $P(X)F(X)P(X)^{-1} = F(0)$ and the matrix of $G(X)$ in the canonical basis of A^s coincides with $F(0)$. From this we conclude that $\langle F(X) \rangle = \text{Im}(F(X)) \cong \text{Im}(G(X)) \cong \text{Im}(i_A) \otimes \text{Im}(F(0)) = A \otimes_R M_0$.

(ii) We have $M \cong A \otimes_R M_0$, for some finitely generated projective R -module M_0 , but by Proposition 2.1, $M_0 \cong M/\langle X \rangle M = \langle F(X) \rangle / \langle X \rangle \langle F(X) \rangle = \langle F(0) \rangle$, so M_0 is generated by s elements. Thus, we have $\text{Im}(F(X)) = \langle F(X) \rangle \cong A \otimes_R \langle F(0) \rangle = \text{Im}(G(X))$, where $F(X)$ and $G(X) = F(0)$ are the idempotent endomorphisms of A^s as in (i). Let $H(X) : \text{Im}(F(X)) \rightarrow \text{Im}(F(0))$ be an isomorphism. We have

$$A^s = \text{Im}(F(X)) \oplus \ker(F(X)) = \text{Im}(F(0)) \oplus \ker(F(0)).$$

Let $T(X)$ be any A -homomorphism from $\ker(F(X))$ to $\ker(F(0))$, for example, $\mathbf{w}(X)T(X) := \mathbf{w}(0)$, with $\mathbf{w}(X) \in \ker(F(X))$. We have the following diagram

$$\begin{array}{ccc} A^s & \xrightarrow{F(X)} & A^s \\ P(X) \downarrow & & \downarrow P(X) \\ A^s & \xrightarrow{F(0)} & A^s \end{array}$$

where $P(X)$ is the A -homomorphism defined by $P(X) := H(X) \oplus T(X)$. Note that the diagram is commutative: If $\mathbf{v}(X) = \mathbf{u}(X)F(X) + \mathbf{w}(X)$, with $\mathbf{u}(X) \in A^s$ and $\mathbf{w}(X) \in \ker(F(X))$, then

$$\mathbf{v}(X)F(X)P(X) = \mathbf{u}(X)F(X)^2P(X) + \mathbf{w}(X)F(X)P(X) = \mathbf{u}(X)F(X)H(X);$$

on the other side,

$$\mathbf{v}(X)P(X)F(0) = [\mathbf{u}(X)F(X)H(X) + \mathbf{w}(X)T(X)]F(0) = \mathbf{u}(X)F(X)H(X)F(0) + \mathbf{w}(X)T(X)F(0) = \mathbf{u}(X)F(X)H(X),$$

where the last equality follows from the fact that $\mathbf{u}(X)F(X)H(X) \in \text{Im}(F(0))$ and $\mathbf{w}(X)T(X) \in \ker(F(0))$.

This proves that $P(X)F(0) = F(X)P(X)$. If $F(X) \neq 0$, then $H(X) \neq 0$ and hence $P(X) \neq 0$; if $F(X) = 0$, then $F(0) = 0$, $H(X) = 0$, $\ker(F(X)) = A^s = \ker(F(0))$ and we can take $T(X) = P(X) = i_{A^s}$.

(iii) is a direct consequence of (i) and (iv) follows from (ii). \square

Remark 2.4. In the proof of (ii) we observed that

$$\ker(F(X)) \cong A^s / \text{Im}(F(X)) \text{ and } \ker(F(0)) \cong A^s / \text{Im}(F(0))$$

are finitely presented modules with $\text{Im}(F(X)) \cong \text{Im}(F(0))$. If there exists at least one surjective homomorphism $T(X)$ from $\ker(F(X))$ to $\ker(F(0))$ and A is left Noetherian, then $P(X)$ is surjective, and hence bijective, i.e., in this situation $F(X) \approx F(0)$.

The following results relate the extended condition \mathcal{E} and the \mathcal{PF} condition for the Ore extension A .

Proposition 2.5. *Suppose that R is \mathcal{PF} . A is \mathcal{E} if and only if A is \mathcal{PF} .*

Proof. \Rightarrow): Let M be a f.g. projective A -module, then $M \cong A \otimes_R M_0$, where M_0 is a f.g. projective R -module (Proposition 2.1). But since R is \mathcal{PF} , then M_0 is R -free and hence M is A -free.

\Leftarrow): If M is a f.g. projective left A -module, then M is A -free, then by Proposition 2.1, M is extended from R . \square

Proposition 2.6. *If A is \mathcal{PF} , then R is \mathcal{PF} .*

Proof. We know that $A/\langle X \rangle \cong R$, so the result follows from Proposition 1.3. \square

Corollary 2.7. *A is \mathcal{PF} if and only if R is \mathcal{PF} and A is \mathcal{E}*

Proof. This is direct consequence of Propositions 2.5 and 2.6. \square

Corollary 2.8. *Let R be a Noetherian, regular and \mathcal{PSF} ring. Then, A is \mathcal{H} if and only if R is \mathcal{PF} and A is \mathcal{E} .*

Proof. This follows from [8] Corollary 2.8, and Corollary 2.7. \square

Remark 2.9. (i) Let $A := R[x; \sigma]$ be the skew polynomial ring over $R = K[y]$, where K is a field and $\sigma(y) := y + 1$. From [10] 12.2.11 we know that A is not \mathcal{E} with respect to R , and hence, from numeral (ii) in Proposition 1.3, we conclude that A is not \mathcal{E} with respect to K . But precisely observe that $A = K[t; i_K][x; \sigma]$ is an Ore extension such that the σ 's are different and $tx \neq xt$. Thus, the conditions we are assuming in this paper about the commutativity of the variables and the restriction to only one automorphism for the ring of coefficients are more than important.

(ii) On the other hand, since R is a commutative principal ideal domain (PID) then R is \mathcal{PF} , therefore, by Corollary 2.7, A is not \mathcal{PF} . This means that in Theorem 5.2 below we can not weak the condition on K to be a commutative PID .

(iii) In addition, observe that R is a commutative Noetherian regular ring with finite Krull dimension and however A is not \mathcal{E} , so the Bass-Quillen conjecture (see [7]) in the case of our Ore extensions conduces to Quillen-Suslin Theorem 5.2.

(iv) Finally, this example also shows that although R is \mathcal{H} , $R[x; \sigma]$ is not \mathcal{H} . In fact, since R is \mathcal{PF} , we conclude that R is \mathcal{PSF} , so the claimed follows from Corollary 2.8. Thus, the Hermite conjecture for Ore extensions fails (see [7]).

3 Varsenstein's theorem

Let R be a commutative ring, the Vaserstein's theorem in commutative algebra says that if $F(x_1, \dots, x_n) \in M_{r \times s}(R[x_1, \dots, x_n])$, then, $F(x_1, \dots, x_n) \sim F(0)$ if and only if for every $\mathfrak{m} \in \text{Max}(R)$, $\overline{F(x_1, \dots, x_n)} \sim \overline{F(0)}$, where $\overline{F(x_1, \dots, x_n)}$ represents the image of $F(x_1, \dots, x_n)$ in $R_{\mathfrak{m}}[x_1, \dots, x_n]$ and \sim denotes the relation of equivalence between matrices, i.e.,

$$F(X) = P(X)F(0)Q(X),$$

with $P(X) \in GL_r(R[x_1, \dots, x_n])$ and $Q(X) \in GL_s(R[x_1, \dots, x_n])$. In this section we extends this theorem to Ore extensions of type $A := R[x_1, \dots, x_n; \sigma]$.

Recall (see [9]) that if S is a multiplicative system of R and $\sigma(S) \subseteq S$, then $S^{-1}A$ exists and

$$S^{-1}A \cong (S^{-1}R)[x_1, \dots, x_n; \bar{\sigma}], \text{ with } \bar{\sigma}\left(\frac{r}{s}\right) := \frac{\sigma(r)}{\sigma(s)}.$$

In particular, if $\mathfrak{m} \in \text{Max}(R)$ and $S := R - \mathfrak{m}$, we write

$$A_{\mathfrak{m}} := S^{-1}A \cong R_{\mathfrak{m}}[x_1, \dots, x_n; \bar{\sigma}], \text{ where } \sigma \text{ satisfies } \sigma(s) \notin \mathfrak{m} \text{ for any } s \notin \mathfrak{m}.$$

From now on in the present paper we will assume that σ satisfies the following condition:

♣: Given $\mathfrak{m} \in \text{Max}(R)$, if $s \notin \mathfrak{m}$, then $\sigma(s) \notin \mathfrak{m}$.

Some preliminary results are needed for the main theorem.

Proposition 3.1. *Let B be a ring and σ an endomorphism of B . Then,*

- (i) *For every $r \geq 1$, $M_r(B[x_1, \dots, x_n; \sigma]) \cong M_r(B)[x_1, \dots, x_n; \sigma]$.*
- (ii) *If $\sigma(Z(B)) \subseteq Z(B)$ and $s \in Z(B)$, then*

$$\begin{aligned} \phi : B[x_1, \dots, x_n; \sigma] &\rightarrow B[x_1, \dots, x_n; \sigma] \\ p(x_1, \dots, x_n) &\mapsto p(sx_1, \dots, sx_n) \end{aligned}$$

is a ring homomorphism.

(iii) φ defined as

$$\varphi : B[x_1, \dots, x_n; \sigma] \rightarrow B[x_1, \dots, x_n; y_1, \dots, y_n; \sigma] , \varphi(p(x_1, \dots, x_n)) := p(x_1 + y_1, \dots, x_n + y_n)$$

is a ring homomorphism.

Proof. (i) Using an inductive argument we only need to show that $M_r(B[x_1; \sigma]) \cong M_r(B)[x_1; \sigma]$. If we define $\sigma(F) := [\sigma(f_{ij})]$, with $F := [f_{ij}] \in M_r(B)$, then the claimed isomorphism is given by

$$F^{(0)} + F^{(1)}x_1 + \dots + F^{(t)}x_1^t \mapsto [\sum_{k=0}^t f_{ij}^{(k)} x_1^k].$$

(ii) It is clear that ϕ is additive and $\phi(1) = 1$. So, we have to show that $\phi(ax^\alpha bx^\beta) = \phi(ax^\alpha)\phi(bx^\beta)$ for every $a, b \in B$ and $\alpha, \beta \in \mathbb{N}^n$. Since $\sigma^k(s) \in Z(B)$ for every $k \geq 0$, then

$$\begin{aligned} \phi(ax^\alpha bx^\beta) &= \phi(a\sigma^\alpha(b)x^{\alpha+\beta}) = a\sigma^\alpha(b)(sx_1)^{\alpha_1+\beta_1} \dots (sx_n)^{\alpha_n+\beta_n} \\ &= a\sigma^\alpha(b)s\sigma(s)\sigma^2(s) \dots \sigma^{\alpha_1+\alpha_2+\dots+\alpha_n+\beta_1+\beta_2+\dots+\beta_n-1}(s)x^{\alpha+\beta}; \\ \phi(ax^\alpha)\phi(bx^\beta) &= a(sx_1)^{\alpha_1} \dots (sx_n)^{\alpha_n} b(sx_1)^{\beta_1} \dots (sx_n)^{\beta_n} \\ &= a\sigma^\alpha(b)s\sigma(s)\sigma^2(s) \dots \sigma^{\alpha_1+\alpha_2+\dots+\alpha_n+\beta_1+\beta_2+\dots+\beta_n-1}(s)x^{\alpha+\beta}. \end{aligned}$$

(iii) Obviously φ is additive and $\varphi(1) = 1$. Only rest to show that $\varphi(ax^\alpha bx^\beta) = \varphi(ax^\alpha)\varphi(bx^\beta)$ for every $a, b \in B$ and $\alpha, \beta \in \mathbb{N}^n$.

$$\begin{aligned} \varphi(ax^\alpha bx^\beta) &= \varphi(a\sigma^\alpha(b)x^{\alpha+\beta}) = a\sigma^\alpha(b)(x_1 + y_1)^{\alpha_1+\beta_1} \dots (x_n + y_n)^{\alpha_n+\beta_n}; \\ \varphi(ax^\alpha)\varphi(bx^\beta) &= a(x_1 + y_1)^{\alpha_1} \dots (x_n + y_n)^{\alpha_n} b(x_1 + y_1)^{\beta_1} \dots (x_n + y_n)^{\beta_n} \\ &= a\sigma^\alpha(b)(x_1 + y_1)^{\alpha_1+\beta_1} \dots (x_n + y_n)^{\alpha_n+\beta_n}. \end{aligned}$$

□

Lemma 3.2. Let B be a ring, $S \subset Z(B)$ a multiplicative system of B . Let $B[x_1, \dots, x_n; \sigma]$ be an Ore extension such that $\sigma(Z(B)) \subseteq Z(B)$. Given the matrices $F(X) \in M_{r \times s}(B[x_1, \dots, x_n; \sigma])$, $G(X) \in M_{s \times t}(B[x_1, \dots, x_n; \sigma])$ and $H(X) \in M_{r \times t}(B[x_1, \dots, x_n; \sigma])$, let $\overline{L(X)}$ be the image of the matrix $L(X)$ corresponding to the canonical homomorphism

$$B[x_1, \dots, x_n; \sigma] \rightarrow (S^{-1}B)[x_1, \dots, x_n; \overline{\sigma}], \quad \overline{\sigma}\left(\frac{r}{s}\right) := \frac{\sigma(r)}{\sigma(s)}.$$

Suppose that $\overline{F(X)G(X)} = \overline{H(X)}$ and $F(0)G(0) = H(0)$. Then, there exists $s \in S$ such that $F(sX)G(sX) = H(sX)$, where $L(sX) := L(sx_1, \dots, sx_n)$.

Proof. Let $D(X) := F(X)G(X) - H(X)$; since $F(0)G(0) - H(0) = 0$ then

$$D(X) = D^{(1)}x^{\alpha_1} + D^{(2)}x^{\alpha_2} + \dots + D^{(\ell)}x^{\alpha_\ell},$$

with $D^{(k)} \in M_{r \times t}(R)$, and $x^{\alpha_k} := x_1^{\alpha_{k1}} \dots x_n^{\alpha_{kn}}$, where $\alpha_{k1} + \dots + \alpha_{kn} > 0$ for every $1 \leq k \leq \ell$.

From $\overline{D(X)} = \overline{F(X)G(X) - H(X)} = \overline{F(X)}\overline{G(X)} - \overline{H(X)} = \overline{0}$ we conclude that

$$\overline{D^{(1)}}x^{\alpha_1} + \overline{D^{(2)}}x^{\alpha_2} + \dots + \overline{D^{(\ell)}}x^{\alpha_\ell} = \overline{0}.$$

Then, each entry $d_{ij}^{(k)}$ of the matrix $D^{(k)}$ is such that $\frac{d_{ij}^{(k)}}{1} = \frac{0}{1}$ in $S^{-1}R$, so we find $s_{ij}^{(k)} \in S$ such that $s_{ij}^{(k)}d_{ij}^{(k)} = 0$. Let $s := \prod_{i,j,k} s_{ij}^{(k)}$, then $s \in Z(B)$ and $D^{(k)}s = 0$ for each $k = 1, \dots, \ell$. Thus, using Lemma 3.1, we get

$$D^{(1)}s\sigma(s)\sigma^2(s)\cdots\sigma^{\alpha_{i1}+\cdots+\alpha_{in}-1}(s)x^{\alpha_1}+\cdots+D^{(\ell)}s\sigma(s)\sigma^2(s)\cdots\sigma^{\alpha_{i1}+\cdots+\alpha_{in}-1}(s)x^{\alpha_\ell}=0.$$

□

Theorem 3.3 (Vaserstein's theorem). *Let R be a commutative ring and $A := R[x_1, \dots, x_n; \sigma]$. Then, $F(X) \in M_{r \times s}(A)$ is equivalent to $F(0)$ if and only if $F(X)$ is locally equivalent to $F(0)$ for every $\mathfrak{m} \in \text{Max } R$.*

Proof. \Rightarrow): Evident.

\Leftarrow): We denote I the set of elements $a \in R$ with the following property:

Given $f = (f_1, \dots, f_n), g = (g_1, \dots, g_n) \in A^n$ with $f - g \in aA^n$, then $F(f) \sim F(g)$.

$F(f)$ represents the evaluation $x_i \mapsto f_i$, $1 \leq i \leq n$, on the matrix $F(X)$. We claim that I is an ideal of R . In fact, let $a, b \in I$ and $f - g \in (a - b)A^n$, then $f - g = (a - b)h$, with $h \in A^n$, so $f - (g - bh) = ah \in aA^n$, and hence $F(f) \sim F(g - bh)$. But $g - (g - bh) = bh \in bA^n$, so $F(g - bh) \sim F(g)$, whence, $a - b \in I$. Let $r \in R$, $a \in I$ and $f - g \in arA^n \subseteq aA^n$, therefore $F(f) \sim F(g)$, and this means that $ar \in I$.

If we show that $I = R$, then for every $f, g \in A^n$, $F(f) \sim F(g)$, in particular, if $f = (x_1, \dots, x_n)$ and $g = (0, \dots, 0)$, we obtain $F(X) \sim F(0)$. Let $\mathfrak{m} \in \text{Max } R$; there exists $\overline{G(X)} \in \text{GL}_r(R_{\mathfrak{m}}[x_1, \dots, x_n; \overline{\sigma}])$ and $\overline{H(X)} \in \text{GL}_s(R_{\mathfrak{m}}[x_1, \dots, x_n; \overline{\sigma}])$ such that

$$\overline{F(X)} = \overline{G(X)} \overline{F(0)} \overline{H(X)}.$$

Introducing the Ore extension $R_{\mathfrak{m}}[x_1, \dots, x_n; y_1, \dots, y_n; \overline{\sigma}]$, i.e., $x_i x_j = x_j x_i$, $x_i y_j = y_j x_i$ and $y_i y_j = y_j y_i$ for $1 \leq i, j \leq n$, and $y_i \frac{r}{s} := \overline{\sigma}(\frac{r}{s}) y_i = \frac{\sigma(r)}{\sigma(s)} y_i$, we obtain, from Proposition 3.1, that

$$\overline{F(X + Y)} = \overline{G(X + Y)} \overline{F(0)} \overline{H(X + Y)}, \text{ where } Y := (y_1, \dots, y_n). \quad (3.1)$$

Since $\overline{F(0)} = \overline{G(X)}^{-1} \overline{F(X)} \overline{H(X)}^{-1}$, we get

$$\overline{F(X + Y)} = \overline{G(X + Y)} \overline{G(X)}^{-1} \overline{F(X)} \overline{H(X)}^{-1} \overline{H(X + Y)}.$$

Denote $G^* := \overline{G(X + Y)} \overline{G(X)}^{-1}$ and $H^* := \overline{H(X)}^{-1} \overline{H(X + Y)}$. Observe that G^* has the form

$$\overline{G_0(X)} + \overline{G_1(X)} y^{\alpha_1} + \cdots + \overline{G_\ell(X)} y^{\alpha_\ell},$$

with $\overline{G_i(X)} \in M_r(R_{\mathfrak{m}}[x_1, \dots, x_n; \overline{\sigma}])$, for every $i = 1, \dots, \ell$, where $\overline{G_0(X)}$ is the identity matrix, and $y^{\alpha_i} := y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}}$, for every $1 \leq i \leq \ell$. Moreover,

$$\overline{G_i(X)} y^{\alpha_i} = \overline{E_0} y^{\alpha_i} + \cdots + \overline{E_{i_j}} y^{\alpha_i} x^{\beta_{i_j}}, \text{ where } \overline{E_k} \in M_r(R_{\mathfrak{m}}), \text{ for } 0 \leq k \leq i_j. \quad (3.2)$$

Taking a common denominator we find $s' \in S$ and matrices $D_k \in M_r(R)$ such that

$$\overline{E_k} = \frac{D_k}{s'} = \frac{D_k \sigma(s') \sigma^2(s') \cdots \sigma^{\alpha_{i1} + \cdots + \alpha_{in} - 1}(s')}{s' \sigma(s') \sigma^2(s') \cdots \sigma^{\alpha_{i1} + \cdots + \alpha_{in} - 1}(s')},$$

so we can assume that $\overline{E_k} = \frac{D_k}{s' \sigma(s') \sigma^2(s') \cdots \sigma^{\alpha_{i1} + \cdots + \alpha_{in} - 1}(s')}$

Hence, replacing Y by $s'Y$ we get that $\overline{G(X + s'Y)} \overline{G(X)}^{-1}$ is the image of a matrix with entries over $R[x_1, \dots, x_n; y_1, \dots, y_n; \sigma]$. In a similar way we can do with $\overline{H(X)}^{-1} \overline{H(X + s'Y)}$. Thus, we can suppose that

$$\overline{G(X + s'Y)} \overline{G(X)}^{-1} \quad \text{and} \quad \overline{H(X)}^{-1} \overline{H(X + s'Y)}$$

are images of invertible matrices

$$\Gamma(X, Y) \quad \text{and} \quad \Delta(X, Y),$$

respectively, with entries in $R[x_1, \dots, x_n; y_1, \dots, y_n; \sigma]$, where $\Gamma(X, 0)$ and $\Delta(X, 0)$ are identities matrices.

On $R_{\mathfrak{m}}[x_1, \dots, x_n; y_1, \dots, y_n; \bar{\sigma}]$, we have the equation

$$\overline{F(X + s'Y)} = \overline{G(X + s'Y)} \overline{G(X)}^{-1} \overline{F(X)} \overline{H(X)}^{-1} \overline{H(X + s'Y)},$$

and on $R[x_1, \dots, x_n; \sigma]$,

$$F(X) = \Gamma(X, 0)F(X)\Delta(X, 0).$$

Taking $B := R[x_1, \dots, x_n; \sigma]$ in Lemma 3.2, there exists $s'' \in R - \mathfrak{m}$ such that for $s := s's''$, we have the equation

$$F(X + sY) = \Gamma(X, s''Y)F(X)\Delta(X, s''Y)$$

in the Ore extension $R[x_1, \dots, x_n; y_1, \dots, y_n; \sigma]$. Now if $f, g, h \in A^n$ are such that $f - g = sh$, we have

$$F(f) = F(g + sh) = \Gamma(g, s''h)F(g)\Delta(g, s''h),$$

where $\Gamma(g, s''h)$ and $\Delta(g, s''h)$ are invertible; then $F(f) \sim F(g)$ and so $s \in I$. We have showed that for every $\mathfrak{m} \in \text{Max } R$ there exists $s \in I$ with $s \notin \mathfrak{m}$, i.e., $I = R$. \square

4 Quillen's patching theorem

Now we will study another classical result of commutative algebra for the Ore extensions of type $A := R[x_1, \dots, x_n; \sigma]$, with R commutative: the famous Quillen's patching theorem. For this we will adapt the method studied in [6]. Let B be a ring and consider two exact sequences of B -modules

$$\begin{aligned} 0 \rightarrow K_1 \xrightarrow{\beta_1} F_1 \xrightarrow{\alpha_1} M_1 \rightarrow 0 \\ 0 \rightarrow K_2 \xrightarrow{\beta_2} F_2 \xrightarrow{\alpha_2} M_2 \rightarrow 0. \end{aligned}$$

where F_1, F_2 are free.

Proposition 4.1. *If $i : M_1 \rightarrow M_2$ is an isomorphism, then there exists $\alpha \in \text{Aut}(F_1 \oplus F_2)$ such that following diagram*

$$\begin{array}{ccc} F_1 \oplus F_2 & \xrightarrow{(\alpha_1, 0)} & M_1 \\ \alpha \downarrow & & \downarrow i \\ F_1 \oplus F_2 & \xrightarrow{(0, \alpha_2)} & M_2 \end{array} \quad (4.1)$$

is commutative. Identifying K_j with $\beta_j(K_j) \subseteq F_j$ ($j = 1, 2$), $\alpha(K_1 \oplus F_2) = F_1 \oplus K_2$.

Proof. The proof is exactly as in [6] and it is not necessary to assume that B is commutative. \square

Corollary 4.2. *Let B be a ring and consider two exact sequences of B -modules*

$$F'_j \xrightarrow{\beta_j} F_j \xrightarrow{\alpha_j} M_j \rightarrow 0 \quad (j = 1, 2), \quad (4.2)$$

where F_j, F'_j are free B -modules. Then, $M_1 \cong M_2$ if and only if there exist $\alpha \in \text{Aut}(F_1 \oplus F_2)$ and $\beta \in \text{Aut}(F'_1 \oplus F_2 \oplus F_1 \oplus F'_2)$ such that the following diagram

$$\begin{array}{ccc} F'_1 \oplus F_2 \oplus F_1 \oplus F'_2 & \xrightarrow{(\beta_1 \oplus i_{F_2}, 0)} & F_1 \oplus F_2 \\ \beta \downarrow & & \downarrow \alpha \\ F'_1 \oplus F_2 \oplus F_1 \oplus F'_2 & \xrightarrow{(0, i_{F_1} \oplus \beta_2)} & F_1 \oplus F_2 \end{array} \quad (4.3)$$

commutes.

Proof. See [6]. □

Remark 4.3. Now we can consider that M_1 and M_2 are finitely presented B -modules

$$\begin{aligned} F'_1 &\xrightarrow{\beta_1} F_1 \xrightarrow{\alpha_1} M_1 \rightarrow 0 \\ F'_2 &\xrightarrow{\beta_2} F_2 \xrightarrow{\alpha_2} M_2 \rightarrow 0; \end{aligned}$$

with respect to the canonical bases, β_1 is given by a matrix $B_1 \in M_{m' \times m}(B)$ and β_2 by $B_2 \in M_{n' \times n}(B)$. The matrices that represent the homomorphisms in the rows of (4.3) are given by

$$\left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & I_n \\ \hline 0 & \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c|c} 0 & \\ \hline I_m & 0 \\ \hline 0 & B_2 \end{array} \right], \quad (4.4)$$

and the matrices of homomorphisms in the columns are in $M_r(B)$ and $M_s(B)$, where $r := m + n + m' + n'$ and $s := m + n$. Therefore, Corollary 4.2 says that the modules M_1 and M_2 are isomorphic if and only if the matrices in (4.4) are equivalent.

Note that $A_{\mathfrak{m}}$ is a right A -module and if M is a left A -module, then we denote

$$M_{\mathfrak{m}} := A_{\mathfrak{m}} \otimes_A M = R_{\mathfrak{m}}[x_1, \dots, x_n; \bar{\sigma}] \otimes_A M.$$

If N is a right R -module, then we denote

$$N[x_1, \dots, x_n; \sigma] := N \otimes_R A.$$

Theorem 4.4 (Quillen's patching theorem). *Let R be a commutative ring and $A := R[x_1, \dots, x_n; \sigma]$. Let M be a finitely presented A -module. M is extended from R if and only if $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$, for every $\mathfrak{m} \in \text{Max}(R)$.*

Proof. There exists an exact sequence of A -modules

$$A^p \xrightarrow{\beta_1} A^q \xrightarrow{\alpha_1} M \longrightarrow 0. \quad (4.5)$$

Tensoring by $A/\langle X \rangle$ we obtain the exact sequence of R -modules

$$R^p \xrightarrow{\bar{\beta}_1} R^q \xrightarrow{\bar{\alpha}_1} M/\langle X \rangle M \longrightarrow 0. \quad (4.6)$$

If $B \in M_{p \times q}(A)$ is the matrix of β_1 with respect to the canonical bases, then $B(0)$ is the matrix of $\bar{\beta}_1$. From (4.6) we get an exact sequence of A -modules, where $N := M/\langle X \rangle M$:

$$R^p[x_1, \dots, x_n; \sigma] \xrightarrow{\bar{\beta}_1[X]} R^q[x_1, \dots, x_n; \sigma] \xrightarrow{\bar{\alpha}_1[X]} N[x_1, \dots, x_n; \sigma] \rightarrow 0. \quad (4.7)$$

Note that $\bar{\beta}_1[X] := \bar{\beta}_1 \otimes i_A$ and $\bar{\alpha}_1[X] := \bar{\alpha}_1 \otimes i_A$. The sequence (4.7) can be identified with the exact sequence of A -modules

$$A^p \xrightarrow{\beta_2} A^q \xrightarrow{\alpha_2} N[x_1, \dots, x_n; \sigma] \longrightarrow 0, \quad (4.8)$$

where β_2 is described by the matrix $B(0)$. From Corollary 4.2, we have $M \cong N[x_1, \dots, x_n; \sigma]$ if and only if the $2(p+q) \times (2q)$ -matrices

$$F := \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I_q \\ \hline 0 & \end{array} \right] \quad \text{and} \quad G := \left[\begin{array}{c|c} 0 & \\ \hline I_q & 0 \\ \hline 0 & B(0) \end{array} \right]$$

are equivalent. Note that G is equivalent to $F(0)$ (permuting rows and columns). Therefore, by Theorem 3.3, $M \cong N[x_1, \dots, x_n; \sigma]$ if and only if F and $F(0)$ are locally equivalent for every $\mathfrak{m} \in \text{Max}(R)$. Since the exact sequences (4.5) and (4.8) are consistent with respect to the localization by \mathfrak{m} , we get that M is extended from R if and only if $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$, for every $\mathfrak{m} \in \text{Max}(R)$. □

Corollary 4.5. *Let R be a commutative ring and $A := R[x_1, \dots, x_n; \sigma]$. If for each $\mathfrak{m} \in \text{Max } R$, $A_{\mathfrak{m}}$ is \mathcal{E} with respect to $R_{\mathfrak{m}}$, then A is \mathcal{E} .*

Proof. Let $M \in \mathfrak{P}(A)$, then $M_{\mathfrak{m}} \in \mathfrak{P}(A_{\mathfrak{m}})$, and hence, by the hypothesis, $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$, for every $\mathfrak{m} \in \text{Max } R$. Using Theorem 4.4 (note that M is finitely presented as A -module), M is extended from R , and hence, A is \mathcal{E} . \square

5 Quillen-Suslin theorem

This last section concerns with Quillen-Suslin's theorem. Here R is non-commutative but some other extra conditions on it are assumed as well as over σ .

Theorem 5.1 (Horrocks' theorem). *Let R be a left regular domain and Z its center. Suppose that Z is Noetherian, R is finitely generated over Z and σ is an automorphism of R of finite order d , with d invertible in Z . Suppose that $P \in \mathfrak{P}(A)$ is stably extended from R and the rank of P is at least 2. Then P is extended from R .*

Proof. Firstly we recall that the rank of P means the maximal number of R -independent elements of P , and P is stably extended from R if there exists $m \geq 0$ such that $P \oplus A^m$ is extended from R .

Denote by d the order of σ . The automorphism σ is mapping Z onto itself. Let Z^σ be the subalgebra in Z of invariants of σ . By Noether's theorem Z^σ is Noetherian and Z is a finitely generated Z^σ -module. We claim that $Z^\sigma[x_1^d, \dots, x_n^d]$ is a central subalgebra in A and A is a finitely generated left $Z^\sigma[x_1^d, \dots, x_n^d]$ -module. In fact monomials in x_1, \dots, x_n commute. Since $x_i r = \sigma(r) x_i$ for any i and for any $r \in R$, then $x_i^d r = \sigma^d(r) x_i^d = r x_i^d$. Hence each element x_i^d is central. Now if $r \in Z^\sigma$ then r commutes with each element of R and with each variable x_i . Since R is a finitely generated Z -module then by Noether's theorem R is a finitely generated Z^σ -module. So the claim is proved. Consider A as a graded ring $A = \bigoplus_n A_n$ where A_n is the span of all monomials of a total degree n . In particular $A_0 = R$. Let V be the graded ideal in $D = Z^\sigma[x_1^d, \dots, x_n^d]$ considered in [2, Theorem 5.32]. Let \wp be a maximal graded ideal in D and A_\wp^+ the localization considered in [2, Definition 5.30]. As it was shown in [2, Corollaries 5.36] either V contains all x_1^d, \dots, x_n^d or the ring A_\wp^+ is a skew polynomial extension $A_\wp^+(0)[x_i, \alpha]$ for some x_i . In the first case if each $x_i^d \in V$ then $P_{x_i^d}$ is extended from R and x_i^d is monic. In the second case using the restriction on the rank of P we can find a monic polynomial f in x_i such that P_f is extended from R . So in both cases by [2, Proposition 5.35] the module P is extended from R . \square

Theorem 5.2 (Quillen-Suslin theorem). *Let K be a field and $A := K[x_1, \dots, x_n; \sigma]$, with σ bijective and having finite order. Then A is \mathcal{PF} .*

Proof. Apply previous theorem with $R = Z = K$. \square

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